

Dynamical complexity of state-conserving cellular automata with a four-cell neighborhood: the three-state case

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Let Q be a finite set of states. The set of all possible configurations of length n is denoted by X_n and is identified with Q^n . The set of all finite configurations (*i.e.*, of all possible length) is denoted by X^* . A one-dimensional cellular automaton (CA) $F : X^* \rightarrow X^*$ with the four-cell neighborhood is determined by a local rule $f : Q^4 \rightarrow Q$ in the following way: if $\mathbf{x} \in X_n$, then $F(\mathbf{x}) \in X_n$ and

$$F(\mathbf{x})_i = f(x_{i-1}, x_i, x_{i+1}, x_{i+2})$$

(here the operations on indices are performed modulo n).

We say that such an automaton is **state-conserving** if for each initial configuration the number of cells in each state is constant throughout the entire evolution of the system.

The case of state-conserving CAs with the three-cell neighborhood was completely characterized in [1] (for all finite state sets).

In the talk, we will present the first results of a broader program aimed at characterizing all one-dimensional state-conserving CAs with four-cell neighborhoods. As an initial step, we investigate in detail the case in which the state set consists of three elements. Particular attention will be devoted to the mathematical framework used

in our analysis, especially techniques involving partially ordered sets and maximal independent sets of suitably defined graphs.

We first obtain a complete classification of all such CAs, thereby confirming the result of [2] that there exist exactly 6312 three-state state-conserving CAs with the four-cell neighborhood. Moreover, we derive a concise structural characterization that not only explains the appearance of this number, but also clarifies several aspects of the global dynamics generated by these automata.

A key feature of the proposed characterization is that it admits a natural extension to larger state sets, providing a promising foundation for the systematic study of more general state-conserving CAs.

Already at this stage, the proposed framework allows us to characterize all state-conserving CAs with the four-cell neighborhood with four states, thereby confirming the result of [2], previously obtained by a brute-force search. Moreover, we were able to determine the number of state-conserving CAs with the four-cell neighborhood with five states (see Table 1), a task that was previously computationally infeasible using brute-force methods.

The next step of the project will be to derive general formulae for the number of state-conserving CAs with the four-cell neighborhood as a function of the number of states, similarly to the case of the three-cell neighborhood studied in [1] (see the column corresponding to $m = 3$ in Table 1). Another important direction is a complete classification of the dynamics generated by such CAs.

$q \backslash m$	2	3	4	5	6	7
2	2	5	22	428	133 184	1 571 814 309
3	2	15	6 312	?	?	?
4	2	89	241 121 850	?	?	?
5	2	843	3 995 550 719 621 580	?	?	?
6	2	11 645	?	?	?	?
7	2	227 895	?	?	?	?
8	2	6 285 809	?	?	?	?

Table 1: Cardinalities of the sets of all state-conserving CAs for different numbers of states q and neighborhood sizes m . Previously known values are taken from [2]; entries shown in red correspond to the new result established by our research team.

The results concerning state-conserving CAs with the four-cell neighborhood in the case of three states can be found at the following address: <https://github.com/mikolajczy01/state-conserving-cellular-automata>.

References

- [1] B. Wolnik, M. Dziemiańczuk, and B. De Baets, “State-conserving one-dimensional cellular automata with radius one,” *Nonlinear Dynamics*, 2025. DOI: [10.1007/s11071-025-10896-9](https://doi.org/10.1007/s11071-025-10896-9).
- [2] É. Miquey, *State-conserving cellular automata*, 2011. [Online]. Available: <https://www.i2m.univ-amu.fr/perso/etienne.miquey/stage/utu.pdf>.